**Commutative Algebra Sheet 1**

$R$ denotes a commutative ring with 1. We assume that $1 \neq 0$. 'Module' will mean '$R$-module'.

1. Let $Y$ be a multiplicatively closed subset of a ring $R$, with $0 \notin Y$. Prove that any ideal $P$ maximal w.r.t. $P \cap Y = \varnothing$ is prime.

2. Let $P_1, \dots, P_n$ be prime ideals in a ring and $I$ any ideal. Show that if $I \subseteq P_1 \cup P_2 \cup \dots \cup P_n$ then $I \subseteq P_j$ for some $j$.

3. Let $\theta : M \to N$ and $\phi : N \to M$ be module homomorphisms such that $\theta \circ \phi = Id_N$, i.e. $\theta\phi(x) = x \ \forall x \in N$. Show that $M = \ker\theta \oplus \phi(N)$.

4. Let $M$ be an extension of $A$ by $B$ (modules), i.e. $A \leq M$ and $M/A \cong B$. (i) Show that if $M$ is finitely generated then so is $B$. (ii) Show that if both $A$ and $B$ are finitely generated then so is $M$. Is the converse true, in general? (iii) Deduce that $M$ is Noetherian if and only both $A$ and $B$ are Noetherian.

5. Let $F$ be a free module with basis $X$. Let $B$ be an arbitrary module and let $\theta : X \to B$ be an arbitrary mapping. Show that there exists a unique module homomorphism $\theta^* : F \to B$ such that $\theta^*(x) = \theta(x)$ for every $x \in X$.

Deduce (i) every module is a homomorphic image of a free module, and an $n$-generator module is a homomorphic image of $R^n$; (ii) if $R$ is Noetherian then every finitely generated $R$-module is Noetherian.

6. Let $F$ be a free module and let $f : M \to F$ be an epimorphism from a module $M$ onto $F$. Show that there exists a homomorphism $h : F \to M$ such that $f \circ h = \mathrm{Id}_F$. (*Hint*: for each $x \in X$, a basis for $F$, choose a pre-image of $x$ under $f$.)

Deduce: If $N < M$ and $M/N$ is free then $N$ has a complement in $M$, i.e. there exists a submodule $C$ of $M$ such that $M = N \oplus C$.

7. (i) Let $M$ be a module. Show that if $M$ is not Noetherian then $M$ has a submodule $N$ such that $N$ is not finitely generated but $A$ is finitely generated whenever $N < A \leq M$.

(ii) Let $R$ be a ring. Prove: if every prime ideal of $R$ is finitely generated then $R$ is Noetherian. [*Hint*: Take $M = R$ in (i). Then $N \geq BC$ where $N < B$ and $N < C$. Note that $B/BC$ is a Noetherian $R/C$-module.]